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On the Calculation of Laminar and Turbulent Boundary Layers on Longitudinally Curved Surfaces

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Introduction

PREVIOUS contributions to the calculation of laminar boundary layers with longitudinal curvature have been made by Narasimha and Ojha,[1] van Tassel and Taulbee,[2] Schultz-Grunow and Breuer,[3] and Murphy[4]; earlier contributions have been reviewed by van Dyke.[5] Most authors have been concerned with isothermal flow (Ref. 2 is a major exception) and only Ref. 1 obtained detailed results for a range of wedge-flow pressure gradients. Turbulent boundary-layer properties have been calculated by So and Mellor,[6] Cebeci,[7] Bradshaw,[8] Rastogi and Whitelaw,[9] and Launder et al.[10] and have made use mainly of mixing-length forms of eddy viscosity. Once again, the emphasis has been on constant-property flow.

As indicated by Schultz-Grunow and Breuer,[3] the laminar boundary-layer equations with constant viscosity, including all second-order terms plus a diffusion term of higher order, may be written as:

$$u\frac{\partial u}{\partial x}+(1+ky)v\frac{\partial u}{\partial y}+kuv=-\frac{1}{\rho}\frac{\partial p}{\partial x}+(1+ky)\nu\frac{\partial^2 u}{\partial y^2}+\nu k\left[\frac{\partial u}{\partial y}-\frac{ku}{(1+ky)}\right] \tag{1}$$

$$ku^2=\frac{(1+ky)}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}(1+ky)v=0 \tag{3}$$

with

$$k=1/R_0$$

where the last term in brackets in Eq. (1) is of third order. Schultz-Grunow and Breuer[3] point out that a second-order treatment of the equations would omit this term, but provide small justification for the omission of other third-order terms which are present when boundary-layer scaling is used in the Navier-Stokes equations. Its inclusion simply allows the boundary condition, Eq. (4), to satisfy analytically the boundary-layer equations, Eqs. (1) and (2), at the "edge" $y=\delta$. By analogy, the same equations apply to turbulent boundary-layer flow with the laminar viscosity replaced by an effective viscosity $\nu+\epsilon_m$. For the present turbulent-flow calculations, the eddy viscosity (ϵ_m) was specified by the standard Cebeci-Smith formulation[11] modified by Bradshaw's correction for longitudinal curvature (see, for example, Ref. 11, Sec. 6.2.6) with a curvature parameter of 3.5.

Equations (1-3) are subject to the following boundary conditions:

$$y=0:\quad u=v=0;\quad y=\delta:\quad u_e=\frac{u_w(x)}{1+ky} \tag{4}$$

where the condition at $y=\delta$ has been obtained assuming $v=0$ and vanishing vorticity in the freestream, and $u_w(x)$ is the inviscid velocity distribution.

The boundary conditions at $y=\delta$ are slightly different from the usual edge boundary condition, namely $u=u_e$, because curvature also enters the condition at the outer edge of the boundary layer. For most boundary layers, transformed coordinates offer advantages and have been used here. By introducing the transformed variables (ξ,η) and a dimensionless stream function $f(\xi,\eta)$, Eqs. (1-4) can be written as:

$$(1+B\eta)(bf'')'+ff''+\frac{B}{1+B\eta}ff'-\frac{B^2}{(1+B\eta)}(bf')+2Bbf''-B(bf')'-\beta f'^2=P+2\xi\left[f'f'_\xi-\left(f''+\frac{B}{1+B\eta}f'\right)f_\xi\right] \tag{5}$$

$$\frac{B}{1+B\eta}f'^2=\frac{\partial p^*}{\partial\eta} \tag{6}$$

$$\eta=0:\quad f=f'=0;\quad \eta\to\eta_\infty:\quad f'\to 1/(1+B\eta) \tag{7}$$

Here, primes denote differentiation with respect to η and

$$B=\frac{k\sqrt{2\xi}}{\rho u_w},\quad \beta=\frac{2\xi}{u_w}\frac{\mathrm{d}u_w}{\mathrm{d}\xi},\quad P=2\xi\frac{\partial p}{\partial\xi}$$

$$p^*=\frac{p}{\rho u_w^2},\quad b=1+\frac{\epsilon_m}{\nu}$$

Note that in Eq. (5) the term $B^2/(1+B\eta)(bf')$ corresponds to the higher-order viscous term of Eq. (1).

Unlike ordinary boundary-layer flows where $f'\to 1$ as $\eta\to\eta_\infty$, the edge boundary condition may be sensitive to the specification of η_∞. If we differentiate the edge boundary condition Eq. (7), then as $\eta\to\eta_\infty$,

$$f''\to -B/(1+B\eta)^2 \tag{8}$$

At first it may seem that the problem is overspecified. However, combining the edge condition, Eqs. (7) and (8), a relation for the outer boundary condition is obtained, namely,

$$\eta\to\eta_\infty\quad f''+B(f')^2=0 \tag{9}$$

Received April 18, 1978; revision received Oct. 23, 1978.

Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Boundary Layers and Convective Heat Transfer—Turbulent.

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Table 1 Comparison of calculated wall shear values for laminar similar flows with u_w = const.

	f_w''			
			Present work	
$B/\sqrt{2}$	Ref. 3	Ref. 1	Eq. (9)	B^2 omitted
−0.05	0.5738	0.5719	0.5744	0.5799
−0.02	0.5109	0.5105	0.5109	0.5095
0	0.4696	0.4696	0.4696	0.4696
0.02	0.4294	0.4287	0.4290	0.4340
0.05	0.3711	0.3673	0.3699	0.3871

The system given by Eqs. (5-7) and (9) is solved with the box method of Keller. The new boundary condition, Eq. (9), adds no complications and is easily accommodated.

Results

Laminar flow calculations were performed to allow comparisons with the previous results of Narasimha and Ojha[1] and Schultz-Grunow and Breuer[3] and to assess the impact of including the higher-order viscous term. Results for a zero-pressure gradient flow were obtained by solving Eq. (5) with and without the third-order viscous term. Inclusion of this term gives solutions which are virtually identical with those of Schultz-Grunow and Breuer (see Table 1). However, the solutions without the third-order term deviate from those of Narasimha and Ojha by as much as 5% at the larger values of curvature used. Since Narasimha and Ojha solved a form of Eq. (5) based on expansion and truncation to a consistent second-order form, it is possible that the second-order form should be limited to values of $B/\sqrt{2} < |0.02|$.

For a laminar, similar, pressure gradient flow ($u_w \alpha x$), again the results from the complete Eq. (5) were in better agreement with those of Ref. 1 than were those obtained by omitting the B^2 term. The influence of the third-order term is, however, small for this pressure gradient (the maximum difference being less than 0.5%), and the close agreement between the present solutions and the second-order consistent equation suggests that the latter is appropriate at values of the curvature parameter, at least to ±0.07. However, it is clear that, in this case, the pressure gradient effect is much greater than that due to curvature.

So and Mellor[6] have conducted experiments which are ideally suited to testing a boundary-layer calculation procedure for turbulent flows on longitudinally curved surfaces. Measurements were obtained in the bounday layer on the convex wall of a channel whose cross section was arranged to result in zero longitudinal pressure gradient over the region of the present calculations. The channel consisted of a straight section 48 in. long followed by a curved wall where the measurements were taken. The calculated results, which were obtained downstream of $x = 54.5$ in., are presented in Figs. 1 and 2 which show velocity profile and the momentum thickness, shape factor, and skin-friction coefficient, respectively. As can be seen the agreement with experiment is excellent except, perhaps, for the skin-friction coefficient where a maximum deviation of around 10% can be seen. Part of this small discrepancy may be attributed to So and Mellor's use of the equation

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = k u_e^2$$

in obtaining their velocity profiles. This equation presumes that $k \ll y$ and, since $\delta/R_0 \simeq 1/11$, gives rise to less full profiles and to higher values of shear stress.

In both this calculation and the one described below, the results presented were obtained by retaining the third-order terms in Eq. (5), since this produced the best comparison in

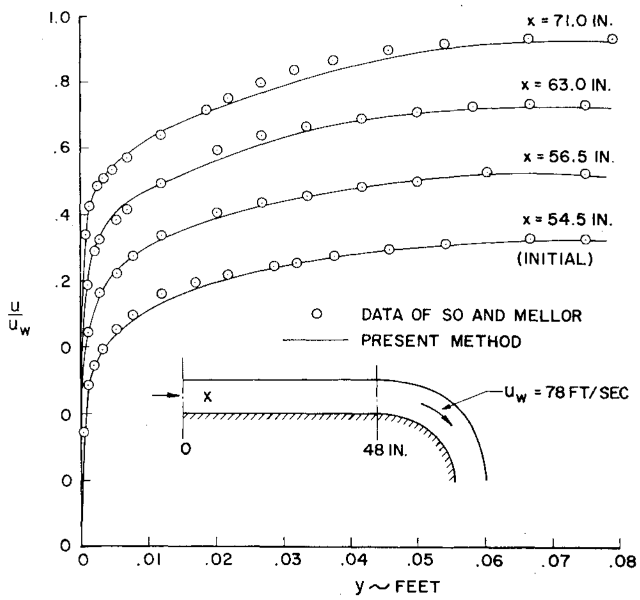

Fig. 1 Comparison of calculated velocity profiles and data of So and Mellor.[6]

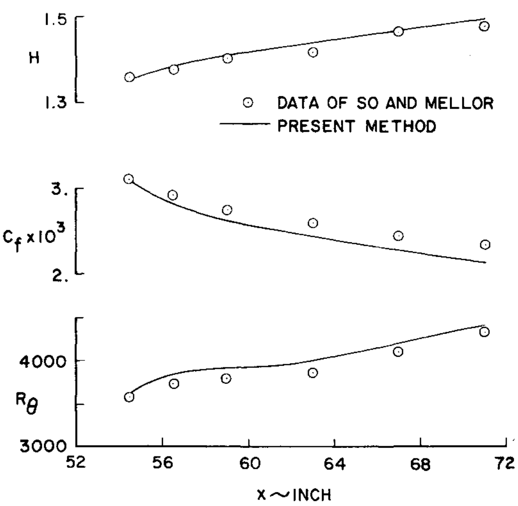

Fig. 2 Comparison of calculated integral thicknesses and skin friction with data of So and Mellor.[6]

laminar flow.Solutions were obtained omitting this term, but the differences in the results were not significant. Since it has already been shown that pressure gradient effects are more important than small amounts of curvature, and the adjustable curvature parameter in the turbulence model is not necessarily fixed at the value we used and can probably be changed to absorb the third-order effect, we conclude that for turbulent flows, either formulation can be used.

Patel[12] has provided measurements for flow in the vicinity of the convex surface of a cylinder, with adverse longitudinal pressure gradient. The flow was tripped 45 deg from the stagnation point and separation was observed at 110 deg. The value of δ/R_0 in this experiment varied from 1/30 at 60 deg to approximately 1/20 at 110 deg. It should be noted that, as pointed out by Bradshaw,[8] δ/R_0 is not entirely representative of the likely effect of curvature. The present calculations began with the measured profile at 60 deg and results are shown in Fig. 3; the two sets of calculated curves represent solutions of the equations with and without the terms which represent curvature. The comparison is inexact, but it is clear that the calculations including curvature effects are in excellent agreement with experiment, although the influence of the curvature is not especially strong until the downstream region.

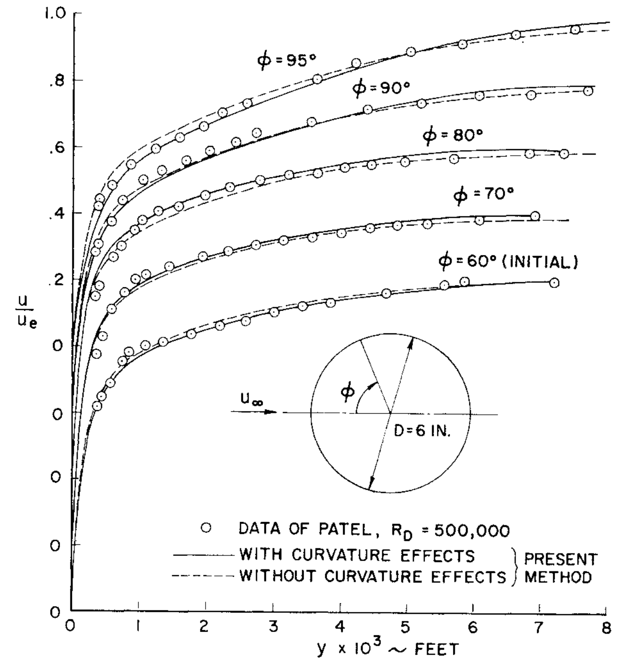

Fig. 3 Comparison of calculated velocity profiles and data of Patel.[12]

The present turbulent flow results have made use of Richardson-number modifications to the eddy viscosity with a value of the curvature parameter, which is of similar order of magnitude to that used in previous investigations.[8] The approximate nature of the equations and physical assumptions insures that there can be no unique value of this parameter which will allow all flows to be represented exactly. With the present equations and the two comparisons made, the value of 3.5 appears to be best, but a single value of this order is likely to produce accurate results over a wide range of flows. The use of higher-order turbulence models is difficult to justify for this purpose, as suggested by Bradshaw.[13] Additional constants are required to represent the curvature in additional conservation equations, which themselves involve significant and untested approximations.

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Subsonic Base Pressure Fluctuations

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Nomenclature

C_{P_b} = base pressure coefficient $C_{P_b} = 2(P_b - P_l)/\rho\, U^2$
D = model diameter
f = frequency
M_l = freestream approach Mach number
P_l = freestream pressure
P_b = base pressure
P_b' = fluctuating component of base pressure
P_{atm} = atmospheric pressure
U = freestream approach velocity
S = Strouhal number, $S = fD/U$
ρ = density

Introduction

THE abrupt change in the rear geometry of a bluff body moving through a real fluid causes the external flow to separate from the body. This separated region, which occurs at or near the base of the body, is usually referred to as the near-wake of the body. The near-wake is dominated by the mixing process associated with the free shear layer which results from the flow separation. While not large, this zone has a significant influence on base drag, base heat transfer, and the configuration of the far-wake. The components which comprise the near-wake flowfield of a blunt axisymmetric body are shown in Fig. 1.

The subsonic base flow problem has been the subject of a great deal of study in recent years. Over the past thirty years, there have been many experimental investigations of the wakes of various axisymmetric bodies and a number of theoretical analyses of the problem. Rather complete reviews of the available literature may be found in Refs. 1 and 2.

One of the most common near-wake parameters reported in the literature is base pressure. In general, the base pressure is presented as a steady-state or time-averaged quantity and little or no mention is made of the base pressure oscillations. However, a knowledge of the base pressure fluctuation is as important as a knowledge of the time-averaged base pressure. The fluctuating component of the base pressure is of prime concern in the consideration of buffet and vibration of blunt-based bodies of revolution. Some data on base pressure fluctuation on various bodies of revolution have been presented by Eldred[3] and Mabey.[4] This Note presents some

Received June 5, 1978, revision received Sept. 25, 1978.

Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions; Aeroacoustics; Viscous Nonboundary-Layer Flows.

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